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Quantum mechanics of complex Hamiltonian systems in one dimension

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Abstract

With a view to obtaining further insight into the nature of eigenvalues and eigenfunctions of a stationary state one-dimensional Schrödinger equation corresponding to a non-Hermitian Hamiltonian $H(x, p)$ we investigate the ground-state solutions for a variety of potentials within the framework of an extended complex phase space characterized by $x = x_1 + \mathrm{i}p_2$, $p = p_1 + \mathrm{i}x_2$, where (x_1, p_1) and (x_2, p_2) are real and considered as canonical pairs. The analyticity property of the eigenfunction alone is found sufficient to throw light on the nature of eigenvalues and eigenfunctions for different systems. It is noted that the imaginary part of the eigenvalue, E_i, turns out to be zero for all potentials $V(x)$ with real couplings whereas it turns out to be nonzero for the case when the couplings are complex. The prescription is also extended to study the excited states. The problems related to the normalization of the eigenfunction and the boundary conditions to be used within this framework are also discussed.

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1. Introduction

In spite of the use of a complex potential in the optical model of the atomic nucleus about 60 years ago [1], the studies of the complex Hamiltonian systems in mathematical terms have not been pursued in the literature to the desired extent. It is only in recent years that such studies have become of considerable interest [2–17, 27, 28] mainly for obtaining a better theoretical understanding of several newly discovered phenomena [18, 19] in different contexts. Further, besides some general studies of complex Hamiltonians in the nonlinear domain [2, 3], efforts have been made to study both classical [4, 5] and quantum [6–17, 27, 28] aspects of the one-dimensional complex Hamiltonian system $H(x, p)$.

With regard to the complexity of $H(x, p)$, it has been introduced and studied in different ways in the literature (for a detailed survey we refer to our earlier work [5]), namely, by

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considering complex couplings in the potential $V(x)$, by complexifying the real coordinate x and the real momentum p through real parameters a and b, namely, $z = ax + \mathrm{i}bp$, $z^* = ax - \mathrm{i}bp$, thus leading to a particular type of complex phase plane. At times, the parameters a and b have also been considered as complex. Another approach to the complex phase space, advocated in recent years [4, 5, 16] and perhaps turning out to be more sound in mathematical terms, can be expressed by writing x and p in the form

$$x = x_1 + \mathrm{i}p_2 \qquad p = p_1 + \mathrm{i}x_2 \tag{1}$$

where the imaginary parts x_2 and p_2 introduced, respectively, in the variables p and x turn out [4] to be canonical pairs such as x_1 and p_1. In the classical context note that $H(x, p)$ now becomes the function of two complex variables and the use of two pairs of Cauchy–Riemann conditions for the analyticity of $H(x, p) = H_1(x_1, p_1, x_2, p_2) + \mathrm{i}H_2(x_1, p_1, x_2, p_2)$, has led [4] to several interesting features regarding the integrability of the associated two-dimensional real systems H_1 and H_2. In the quantum context, on the other hand, since $p \longrightarrow -\mathrm{i}\hbar\frac{\partial}{\partial x}$ which implies $p_1 \longrightarrow -\frac{\partial}{\partial p_2}$, $x_2 \longrightarrow \frac{\partial}{\partial x_1}$, the analyticity of $H(x, p)$ gets translated into that of the complex potential function $V(x)$ and the same is not of immediate concern unless the underlying formalism deals with the derivatives of $V(x)$.

Before proceeding further some pertinent remarks about the non-Hermitian nature of $H(x, p)$ are in order. Firstly, the much studied [6–15] $\mathcal{PJ}$-symmetric Hamiltonians now, in view of the transformation (1), may just correspond to a restriction on the variables x_1, p_1, x_2, p_2, namely, under $\mathcal{PJ}$-symmetry

$$(x_1, p_1, x_2, p_2) \longrightarrow (-x_1, p_1, -x_2, p_2; \mathrm{i} \to -\mathrm{i}).$$

At this stage it should be pointed out that the type of $\mathcal{PJ}$-symmetry with which Bender *et al* [6–8, 25] (as also other authors [10–15, 24, 26]) are dealing is different from the one manifesting in the present approach (cf potential (46)). In fact, the two approaches deal with different types of non-Hermitian Hamiltonians. The non-Hermiticity (or for that matter the $\mathcal{PJ}$-symmetry) arising in the approach of Bender *et al* is mainly due to the complexity of the potential parameters (couplings) whereas in the present case, not only the parameters but also the underlying phase space is considered (cf equation (1)) as complex. One can say that the property of $\mathcal{PJ}$-symmetry of a non-Hermitian Hamiltonian investigated in the present work is of a generalized nature, which, in certain limits (i.e. for the case of real x and p), will reduce to the conventional $\mathcal{PJ}$-symmetry.

Secondly, a transformation similar to (1), which we have used [4, 5] recently following the work of Xavier and de Aguiar [16], was discussed [3] sometime ago by Rao, Buti and Khadkikar (RBK) in the studies of nonlinear evolution equations in the context of amplitude-modulated nonlinear Langmuir waves in plasma. In fact, the type of linkage which we have studied recently [4], between a one-dimensional complex Hamiltonian $H(x, p)$ and the corresponding two, two-dimensional real Hamiltonian systems $H_1(x_1, p_1, x_2, p_2)$ and $H_2(x_1, p_1, x_2, p_2)$, was briefly pointed out by RBK but in a restricted sense and that too without any reference to the Lie Backlund transformation used in [4]. Further note that for the dimensional considerations there appears a constant d in equation (1) in the form $x = x_1 + \mathrm{i}d\,p_2$, $p = p_1 + \mathrm{i}d^{-1}x_2$. In this work, however, we shall choose $d = 1$ for simplicity.

Thirdly, it is well known [20] that the spectral structure of the Korteweg–de Vries (KdV) equation,

$$\frac{\partial U}{\partial t} - 6U\frac{\partial U}{\partial x} + \frac{\partial^3 U}{\partial x^3} = 0 \tag{2}$$

can be obtained, in general, by the Sturm–Liouville equation or, in particular, by the Schrödinger eigenvalue problem, namely,

$$\left[-\frac{\partial^2}{\partial \xi^2} + U(\xi)\right]\psi(\xi) = E\psi(\xi) \tag{3}$$

where $U(\xi)$, acting as a potential term in (3) with the stationary variable $\xi = x - vt$, is a solution of (2). Thus, the complex solutions admitted by the KdV equation (2) can also provide [3] the examples of solvable cases of the corresponding Schrödinger-like equation (3) for complex potentials. We shall return to some of these discussions in section 5. In another case, in the studies of the nonlinear wave–wave interactions Verheest [2] has used the complexity of the Hamiltonian in a different way, i.e. by introducing the complex variables $a_j = \sqrt{J_j}\,\exp(\mathrm{i}\phi_j)$, where J_j and ϕ_j, respectively, are the actions and the angles satisfying the Hamilton equations $\mathring{\phi}_j = (\partial H/\partial J_j)$, $\mathring{J}_j = -(\partial H/\partial \phi_j)$ in the same way as the canonical pairs x and p satisfy. In this case, H turns out to be a function of complex variables a_j and a_j^*.

Finally, a mention may be made of the 'complex scaling' method used by Moiseyev and his co-workers [9]. In this approach one writes the complex-scaled Hamiltonian operator H_θ as

$$H_\theta = S^{-1}(\theta)\hat{H}S(\theta) \tag{4}$$

where x is replaced by $x' = x\exp(-\mathrm{i}\theta)$ and the scale operator S is defined as $S = \exp(\mathrm{i}\theta x\,\mathrm{d}/\mathrm{d}x)$ such that $Sf(x) = f(x\mathrm{e}^{\mathrm{i}\theta})$ for any analytic function $f(x)$. For the quantum system (where $p^2 \longrightarrow -(\partial^2/\partial x^2)$ with $\hbar = m = 1$), however, one obtains

$$H_\theta = -\frac{1}{2}\,\mathrm{e}^{-2\mathrm{i}\theta}\,\frac{\partial}{\partial x'^2} + V(x\,\mathrm{e}^{\mathrm{i}\theta}). \tag{5}$$

In the present work, using the transformation (1) we exploit the analyticity property of the eigenfunction $\psi(x)$ to obtain the solution of the analogous Schrödinger equation

$$\hat{H}(x,p)\,\psi(x) = E\psi(x) \tag{6}$$

where

$$H(x,p) = -\frac{1}{2}\,\frac{\partial^2}{\partial x^2} + V(x) \tag{7}$$

for the complex potential $V(x)$. Note that since equation (6) departs from the conventional conceptual and mathematical setting of the standard [21] Schrödinger equation, we call equation (6) the 'analogous Schrödinger equation' (ASE) for the non-Hermitian operator $H(x,p)$. For this purpose, after using (1) and writing

$$\psi(x) = \psi_{\mathrm{r}}(x_1, p_2) + \mathrm{i}\psi_{\mathrm{i}}(x_1, p_2) \tag{8}$$

we separate [22] equation (6) into a pair of coupled PDEs for ψ_{r} and ψ_{i} and look for their quasi-exact solutions for a variety of potentials using what is known [23] as the 'eigenfunction ansatz method'.

The arrangement of the paper is as follows: in section 2, we carry out the reduction of equation (6) into a pair of coupled PDEs in a quite general manner and look for the ground-state solution of the resultant equations. In section 3, we apply these results to a variety of power, singular and exponential potentials and study the nature of the complex eigenvalue spectra for these potentials. The problems pertaining to the study of excited states and the normalization of the eigenfunction $\psi(x)$ in the extended complex phase space generated by (1) are addressed in section 4. Finally, the findings are summarized and concluding remarks are made in section 5.

2. General results

In order to recast ASE (6) into a pair of coupled PDEs in $\psi_{\rm r}$ and $\psi_{\rm i}$ introduced in equation (8), we also express the complex quantities $V(x)$ and E in the form

$$V(x) = V_{\rm r}(x_1, p_2) + {\rm i}V_{\rm i}(x_1, p_2) \qquad E = E_{\rm r} + {\rm i}E_{\rm i}. \tag{9}$$

In equations (8) and (9) the subscripts r and i, respectively, stand for the real and imaginary parts of the corresponding quantity. Additional subscripts to these quantities separated by a comma will however denote the partial derivatives of the corresponding quantity. Thus, after using (1), (7), (8) and (9) in ASE (6) and separating the real and imaginary parts in the resultant expression, one obtains [22] the following pair of coupled PDEs:

$$-\tfrac{1}{2}\left(\psi_{{\rm r},x_1x_1} - \psi_{{\rm r},p_2p_2} + 2\psi_{{\rm i},x_1p_2}\right) + V_{\rm r}\psi_{\rm r} - V_{\rm i}\psi_{\rm i} = E_{\rm r}\psi_{\rm r} - E_{\rm i}\psi_{\rm i} \tag{10a}$$

$$-\tfrac{1}{2}\left(\psi_{{\rm i},x_1x_1} - \psi_{{\rm i},p_2p_2} + 2\psi_{{\rm r},x_1p_2}\right) + V_{\rm i}\psi_{\rm r} + V_{\rm r}\psi_{\rm i} = E_{\rm r}\psi_{\rm i} + E_{\rm i}\psi_{\rm r}. \tag{10b}$$

Next we use the analyticity property of $\psi(x)$ in terms of the Cauchy–Riemann conditions, namely,

$$\psi_{{\rm r},x_1} = \psi_{{\rm i},p_2} \qquad \psi_{{\rm r},p_2} = -\psi_{{\rm i},x_1} \tag{11}$$

to express equations (10) in somewhat simpler forms as

$$-2\psi_{{\rm r},x_1x_1} + V_{\rm r}\psi_{\rm r} - V_{\rm i}\psi_{\rm i} = E_{\rm r}\psi_{\rm r} - E_{\rm i}\psi_{\rm i} \tag{12a}$$

$$-2\psi_{{\rm i},x_1x_1} + V_{\rm i}\psi_{\rm r} + V_{\rm r}\psi_{\rm i} = E_{\rm r}\psi_{\rm i} + E_{\rm i}\psi_{\rm r} \tag{12b}$$

or

$$E_{\rm r} = V_{\rm r} - \frac{2}{\psi_{\rm i}^2 + \psi_{\rm r}^2}\left[\psi_{\rm r}\,\psi_{{\rm r},x_1x_1} + \psi_{\rm i}\psi_{{\rm i},x_1x_1}\right] \tag{12a'}$$

$$E_{\rm i} = V_{\rm i} - \frac{2}{\psi_{\rm i}^2 + \psi_{\rm r}^2}\left[\psi_{\rm r}\,\psi_{{\rm i},x_1x_1} - \psi_{\rm i}\psi_{{\rm r},x_1x_1}\right]. \tag{12b'}$$

For the (ground-state) solutions of equations (12*a*) and (12*b*) we now make an ansatz, namely,

$$\begin{aligned}\psi(x) \equiv \psi_{\rm r} + {\rm i}\psi_{\rm i} &= \exp(g(x)) \\ &= \exp[g_{\rm r}(x_1, p_2) + {\rm i}g_{\rm i}(x_1, p_2)]\end{aligned} \tag{13}$$

which implies

$$\psi_{\rm r}(x_1, p_2) = \exp[g_{\rm r}(x_1, p_2)]\cos g_{\rm i}(x_1, p_2) \tag{14a}$$

$$\psi_{\rm i}(x_1, p_2) = \exp[g_{\rm r}(x_1, p_2)]\sin g_{\rm i}(x_1, p_2) \tag{14b}$$

where $g_{\rm r}$ and $g_{\rm i}$, in view of conditions (11), satisfy

$$g_{{\rm r},x_1} = g_{{\rm i},p_2} \qquad g_{{\rm r},p_2} = -g_{{\rm i},x_1}. \tag{15}$$

From equations (14*a*) and (14*b*) note that

$$\psi_{\rm r}^2 + \psi_{\rm i}^2 = {\rm e}^{2g_{\rm r}} \qquad \Sigma = \frac{\psi_{\rm i}}{\psi_{\rm r}} = \tan\, g_{\rm i}. \tag{16}$$

Finally, in terms of $g_{\rm r}$ and $g_{\rm i}$, equations (12*a*) and (12*b*) can respectively be expressed as

$$g_{{\rm r},x_1x_1} - g_{{\rm i},x_1}^2 + g_{{\rm r},x_1}^2 + \tfrac{1}{2}(E_{\rm r} - V_{\rm r}) = 0 \tag{17a}$$

$$g_{\mathrm{i},x_1x_1} + 2g_{\mathrm{i},x_1}g_{\mathrm{r},x_1} + \tfrac{1}{2}(E_\mathrm{i} - V_\mathrm{i}) = 0 \tag{17b}$$

or, alternatively, using (15) one obtains

$$g_{\mathrm{i},p_2x_1} - g^2_{\mathrm{r},p_2} + g^2_{\mathrm{i},p_2} + \tfrac{1}{2}(E_\mathrm{r} - V_\mathrm{r}) = 0 \tag{18a}$$

$$g_{\mathrm{r},p_2x_1} + 2g_{\mathrm{r},p_2}\,g_{\mathrm{i},p_2} - \tfrac{1}{2}(E_\mathrm{i} - V_\mathrm{i}) = 0. \tag{18b}$$

Thus, for a given potential $V(x)$ and an ansatz for $\psi(x)$ equations (17*a*) and (17*b*) (or for that matter equations (18*a*) and (18*b*)) can be rationalized to yield the complex eigenvalue E. In the next sections we demonstrate the applications of these results to polynomial, singular and exponential potentials. For the vanishing of the imaginary part of the eigenvalue, E_i (as is the case with $\mathcal{PT}$-symmetric Hamiltonians [7, 8]), it is not difficult to derive from (17*b*) a restriction on the forms of g_r and g_i, namely,

$$g_{\mathrm{i},x_1} = -g_{\mathrm{r},p_2} = \frac{1}{2}\,\mathrm{e}^{-2g_\mathrm{r}}\int \mathrm{e}^{2g_\mathrm{r}}\,V_\mathrm{i}\,\mathrm{d}x_1 + f(p_2) \tag{19}$$

where equations (14)–(16) are used and $f(p_2)$ is some arbitrary function of integration which again can be set equal to zero for simplicity.

3. Applications

3.1. Polynomial potentials

To demonstrate the underlying steps and the intricacies involved in the method, we first take up the well-known case of a complex oscillator and then consider other more complicated forms of the complex polynomial potentials, namely, the quartic potentials.

3.1.1. Complex oscillator. In this case the potential

$$V(x) = ax^2 \qquad (a \text{ real}) \tag{20}$$

using (1), can be expressed as

$$V_\mathrm{r}(x_1, p_2) = a\left(x_1^2 - p_2^2\right) \qquad V_\mathrm{i}(x_1, p_2) = 2ax_1p_2$$

and the ansatz for g_r and g_i, in conformity with (15), turns out to be

$$g_\mathrm{r}(x_1, p_2) = \tfrac{1}{2}\alpha\left(x_1^2 - p_2^2\right) + \beta x_1p_2 \qquad g_\mathrm{i}(x_1, p_2) = \tfrac{1}{2}\beta\left(-x_1^2 + p_2^2\right) + \alpha x_1p_2 \tag{21}$$

where α and β are real. Now using (21) in (17*a*) one arrives at the expression

$$\alpha - (-\beta x_1 + \alpha p_2)^2 + (\alpha x_1 + \beta p_2)^2 + \tfrac{1}{2}E_\mathrm{r} - \tfrac{1}{2}a\left(x_1^2 - p_2^2\right) = 0 \tag{22}$$

which can be rationalized to yield the following relations:

$$E_\mathrm{r} = -2\alpha \tag{23a}$$

$$\beta\alpha = 0 \tag{23b}$$

$$-\beta^2 + \alpha^2 - \tfrac{1}{2}a = 0. \tag{23c}$$

Note, from (23*b*), that for an acceptable solution either $\alpha = 0$ or $\beta = 0$. If $\alpha = 0$, then equation (23*c*) suggests an imaginary value of β which is contrary to ansatz (21). On the other hand, if $\beta = 0$, then $\alpha = \pm\sqrt{a/2}$ and equation (23*a*) leads to

$$E_\mathrm{r} = +\sqrt{2a} \tag{24}$$

for the negative sign in α.

Similarly, if one rationalizes equation (17*b*) using ansatz (21), then one obtains the relations

$$E_{\rm i} = 2\beta \tag{25a}$$

$$-2\beta^2 + 2\alpha^2 - a = 0 \tag{25b}$$

$$\alpha\beta = 0. \tag{25c}$$

Here, while the last two equations turn out to be the same as equations (23*b*) and (23*c*), the consistent values of α and β from these two sets of relations thus lead to $E_{\rm i} = 0$ from (25*a*) and the eigenfunction $\psi(x)$ from (21) and (13) turns out to be [22]

$$\psi(x_1, p_2) = \exp\left[-\frac{1}{2}\sqrt{\frac{a}{2}}\left(x_1^2 - p_2^2 + 2{\rm i}\, x_1 p_2\right)\right]. \tag{26}$$

Next we provide results for some variants of the potential (20).

Case 1. When the parameter a $(=a_{\rm r} + {\rm i}a_{\rm i})$ in (20) becomes complex, then the real and imaginary parts of $V(x)$ can be written as

$$V_{\rm r}(x_1, p_2) = a_{\rm r}\left(x_1^2 - p_2^2\right) - 2a_{\rm i}x_1 p_2 \qquad V_{\rm i}(x_1, p_2) = a_{\rm i}\left(x_1^2 - p_2^2\right) + 2a_{\rm r}x_1 p_2. \tag{27}$$

In this case, using ansatz (21), the rationalization of equations (17*a*) and (17*b*) yields the following consistent values of α and β:

$$\alpha = \pm\tfrac{1}{2}[|a| + a_{\rm r}]^{1/2} \qquad \beta = \pm\tfrac{1}{2}a_{\rm i}[|a| + a_{\rm r}]^{-1/2}$$

and correspondingly the real and imaginary parts of the eigenvalue as

$$E_{\rm r} = [a_{\rm r} + |a|]^{1/2} \qquad E_{\rm i} = [-a_{\rm r} + |a|]^{1/2} \tag{28}$$

where the negative sign in α and the positive sign in β are retained and $|a| = \left(a_{\rm r}^2 + a_{\rm i}^2\right)^{1/2}$. Further, the eigenfunction $\psi(x_1, p_2)$ turns out to be

$$\psi(x_1, p_2) = \exp\left[-\frac{1}{4}\frac{(a + |a|)}{(a_{\rm r} + |a|)^{1/2}}\left(x_1^2 - p_2^2 + 2{\rm i}\, x_1 p_2\right)\right]. \tag{29}$$

Note that the imaginary part of E in this case turns out to be nonzero.

Case 2. Here, we consider the case of a shifted complex oscillator, namely,

$$V(x) = ax^2 + bx \qquad (a, b\,\text{complex}). \tag{30}$$

The ansatze for $g_{\rm r}$ and $g_{\rm i}$ used in this case and consistent with conditions (15) are

$$\begin{aligned} g_{\rm r}(x_1, p_2) &= \tfrac{1}{2}\alpha_{11}\left(x_1^2 - p_2^2\right) + \beta_{11}x_1 p_2 + \alpha_{01}x_1 - \alpha_{10}p_2 \\ g_{\rm i}(x_1, p_2) &= \tfrac{1}{2}\beta_{11}\left(-x_1^2 + p_2^2\right) + \alpha_{11}x_1 p_2 + \alpha_{10}x_1 + \alpha_{01}p_2 \end{aligned} \tag{31}$$

where the constants α_{ij} and β_{ij} are real. These forms of $g_{\rm r}$ and $g_{\rm i}$, when used in (17*a*) and (17*b*), after rationalization immediately yield a set of relations involving $E_{\rm r}$, $E_{\rm i}$ and the parameters α_{ij}, β_{ij}. The same can be solved to give

$$\alpha_{11} = \pm\tfrac{1}{2}\,a_+ \qquad \beta_{11} = \mp\tfrac{1}{2}\,a_-$$

$$\alpha_{10} = \pm\frac{1}{4|a|}\,[b_{\rm i}a_+ - b_{\rm r}a_-] \qquad \alpha_{01} = \pm\frac{1}{4|a|}\,[b_{\rm i}a_- + b_{\rm r}a_+].$$

where $a_+ = (a_{\rm r} + |a|)^{1/2}$, $a_- = (-a_{\rm r} + |a|)^{1/2}$ and $|a| = \left(a_{\rm r}^2 + a_{\rm i}^2\right)^{1/2}$. Finally, the eigenvalue and eigenfunction for the potential (30) are obtained as

$$E_{\mathrm{r}} = \mp a_{+} + \frac{a_{\mathrm{r}}}{4|a|^2}\left(b_{\mathrm{i}}^2 - b_{\mathrm{r}}^2\right) \tag{32a}$$

$$E_{\mathrm{i}} = \mp a_{-} - \frac{1}{4|a|^2}\left[a_{\mathrm{r}} b_{\mathrm{r}} b_{\mathrm{i}} + 2a_{\mathrm{i}}\left(b_{\mathrm{i}}^2 - b_{\mathrm{r}}^2\right)\right] \tag{32b}$$

$$\begin{aligned}\psi(x_1, p_2) &= \exp[g_{\mathrm{r}}(x_1, p_2) + \mathrm{i} g_{\mathrm{i}}(x_1, p_2)] \\ &= \exp\left[\pm\frac{1}{2}\left(Ax^2 + \frac{A^* b}{|a|}x\right)\right]\end{aligned} \tag{33}$$

where $b = b_{\mathrm{r}} + \mathrm{i} b_{\mathrm{i}}$, $A = \frac{1}{2}\left[(a_{\mathrm{r}} + |a|)^{1/2} + \mathrm{i}(-a_{\mathrm{r}} + |a|)^{1/2}\right]$ and $x = x_1 + \mathrm{i} p_2$ from (1) is used.

3.1.2. Complex quartic potential. In this case we consider a quartic potential of very general nature, namely,

$$V(x) = a + bx + cx^2 + dx^3 + ex^4 \qquad (a, b, c, d, e \text{ complex}) \tag{34}$$

or, equivalently,

$$\begin{aligned}V_{\mathrm{r}}(x_1, p_2) &= a_{\mathrm{r}} + b_{\mathrm{r}} x_1 - b_{\mathrm{i}} p_2 + c_{\mathrm{r}}\left(x_1^2 - p_2^2\right) - 2c_{\mathrm{i}} x_1 p_2 + d_{\mathrm{r}}\left(x_1^3 - 3x_1 p_2^2\right) \\ &\quad - d_{\mathrm{i}}\left(3x_1^2 p_2 - p_2^3\right) + e_{\mathrm{r}}\left(x_1^4 - 6x_1^2 p_2^2 + p_2^4\right) - e_{\mathrm{i}}\left(4x_1^3 p_2 - 4x_1 p_2^3\right) \\ V_{\mathrm{i}}(x_1, p_2) &= a_{\mathrm{i}} + b_{\mathrm{r}} p_2 + b_{\mathrm{i}} x_1 + 2c_{\mathrm{r}} x_1 p_2 + c_{\mathrm{i}}\left(x_1^2 - p_2^2\right) + d_{\mathrm{r}}\left(3x_1^2 p_2 - p_2^3\right) \\ &\quad + d_{\mathrm{i}}\left(x_1^3 - 3x_1 p_2^2\right) + e_{\mathrm{r}}\left(4x_1^3 p_2 - 4x_1 p_2^3\right) + e_{\mathrm{i}}\left(x_1^4 - 6x_1^2 p_2^2 + p_2^4\right).\end{aligned}$$

For the ansatz of the eigenfunction we now choose

$$\begin{aligned}g_{\mathrm{r}}(x_1, p_2) &= \beta_{10} x_1 + \beta_{01} p_2 + \beta_{20} x_1^2 + \beta_{02} p_2^2 + \beta_{11} x_1 p_2 + \beta_{30} x_1^3 + \beta_{03} p_2^3 \\ &\quad + \beta_{21} x_1^2 p_2 + \beta_{12} x_1 p_2^2\end{aligned} \tag{35a}$$

$$\begin{aligned}g_{\mathrm{i}}(x_1, p_2) &= \alpha_{10} x_1 + \alpha_{01} p_2 + \alpha_{20} x_1^2 + \alpha_{02} p_2^2 + \alpha_{11} x_1 p_2 + \alpha_{30} x_1^3 + \alpha_{03} p_2^3 \\ &\quad + \alpha_{21} x_1^2 p_2 + \alpha_{12} x_1 p_2^2\end{aligned} \tag{35b}$$

in which the analyticity conditions (15) suggest that

$$3\beta_{30} = \alpha_{21} = -\beta_{12} \qquad \beta_{20} = -2\beta_{02} = \alpha_{11} \qquad \beta_{01} = -\alpha_{10} \qquad \beta_{10} = \alpha_{01}$$

$$2\alpha_{02} = -2\alpha_{20} = \beta_{11} \qquad \beta_{12} = 3\alpha_{03} = -\alpha_{21} \qquad \beta_{21} = -3\beta_{03} = \alpha_{12} = -3\alpha_{30}$$

and the same in turn lead equations (35) to the forms

$$\begin{aligned}g_{\mathrm{r}}(x_1, p_2) &= \alpha_{01} x_1 - \alpha_{10} p_2 + \tfrac{1}{2}\alpha_{11} x_1^2 - \tfrac{1}{2}\alpha_{11} p_2^2 + \beta_{11} x_1 p_2 + \tfrac{1}{3}\alpha_{21} x_1^3 \\ &\quad - \tfrac{1}{3}\alpha_{12} p_2^3 - \alpha_{21} x_1 p_2^2 + \alpha_{12} x_1^2 p_2\end{aligned} \tag{36a}$$

$$\begin{aligned}g_{\mathrm{i}}(x_1, p_2) &= \alpha_{10} x_1 + \alpha_{01} p_2 - \tfrac{1}{2}\beta_{11} x_1^2 + \tfrac{1}{2}\beta_{11} p_2^2 + \alpha_{11} x_1 p_2 - \tfrac{1}{3}\alpha_{12} x_1^3 \\ &\quad - \tfrac{1}{3}\alpha_{21} p_2^3 + \alpha_{12} x_1 p_2^2 + \alpha_{21} x_1^2 p_2.\end{aligned} \tag{36b}$$

As before, the use of these forms of g_{r} and g_{i} in equations (17*a*) and (17*b*), after the rationalization of the resultant expressions, yields the following set of non-repeating equations:

$$E_{\mathrm{r}} = a_{\mathrm{r}} - 2\alpha_{11} + 2\left(\alpha_{10}^2 - \alpha_{01}^2\right) \tag{37a}$$

$$\alpha_{12}\alpha_{21} = -\tfrac{1}{4} e_{\mathrm{i}} \tag{37b}$$

$$\alpha_{21}\beta_{11} + \alpha_{11}\alpha_{12} = -\tfrac{1}{4}d_{\rm i} \tag{37c}$$

$$\alpha_{21}\alpha_{10} - \alpha_{12}\alpha_{01} + \beta_{11}\alpha_{11} = \tfrac{1}{4}\, c_{\rm i} \tag{37d}$$

$$-\alpha_{12} + \alpha_{11}\alpha_{10} - \beta_{11}\alpha_{01} = \tfrac{1}{4}\, b_{\rm i} \tag{37e}$$

$$\alpha_{12}\beta_{11} - \alpha_{21}\alpha_{11} = -\tfrac{1}{4}\, d_{\rm r} \tag{37f}$$

$$\alpha_{21} + \beta_{11}\,\alpha_{10} + \alpha_{11}\alpha_{01} = \tfrac{1}{4}\, b_{\rm r} \tag{37g}$$

$$\left(\alpha_{11}^2 - \beta_{11}^2\right) + 2(\alpha_{01}\alpha_{21} + \alpha_{10}\alpha_{12}) = \tfrac{1}{2}\, c_{\rm r} \tag{37h}$$

$$\alpha_{21}^2 - \alpha_{12}^2 = \tfrac{1}{2}\, e_{\rm r} \tag{37i}$$

$$E_{\rm i} = a_{\rm i} + 2\beta_{11} - 4\alpha_{10}\alpha_{01}. \tag{37j}$$

Here, while equations (37*d*) and (37*h*) will provide the constraining relations among the potential parameters, the pairs of equations [(37*b*), (37*i*)], [(37*c*), (37*f*)] and [(37*e*), (37*g*)] can be immediately solved for the six arbitrary constants in ansatz (36), namely, for $\alpha_{12}, \alpha_{21}, \alpha_{11}, \beta_{11}, \alpha_{10}, \alpha_{01}$. The results thus obtained are

$$\alpha_{21} = \pm\tfrac{1}{2}\, e_+ \qquad \alpha_{12} = \mp\tfrac{1}{2}\, e_- \tag{38}$$

$$\beta_{11} = \pm\frac{1}{4|e|}[d_{\rm r}\, e_- - d_{\rm i}\, e_+] \qquad \alpha_{11} = \pm\frac{1}{4|e|}[d_{\rm r}e_+ + d_{\rm i}\, e_-] \tag{39}$$

$$\alpha_{10} = \pm\frac{1}{2|d|^2}[(b_{\rm i}d_{\rm r} - b_{\rm r}d_{\rm i})\, e_+ + (b_{\rm r}d_{\rm r} + b_{\rm i}d_{\rm i})\, e_- \mp 4(d_{\rm r}e_{\rm i} - d_{\rm i}e_{\rm r})] \tag{40a}$$

$$\alpha_{01} = \pm\frac{1}{2|d|^2}[(b_{\rm r}d_{\rm r} + b_{\rm i}d_{\rm i})\, e_+ + (b_{\rm r}d_{\rm i} - b_{\rm i}d_{\rm r})\, e_- \mp 4(d_{\rm r}e_{\rm r} + d_{\rm i}e_{\rm i})] \tag{40b}$$

where $e_+ = (|e| + e_{\rm r})^{1/2}$ and $e_- = (|e| - e_{\rm r})^{1/2}$. The constraining relations obtained from equations (37*d*) and (37*h*) are given by

$$|d|^2\left[e_{\rm i}\left(d_{\rm r}^2 - d_{\rm i}^2\right) - 2e_{\rm r}d_{\rm r}d_{\rm i}\right] - 8|e|^2[e_{\rm r}(b_{\rm i}d_{\rm r} - b_{\rm r}d_{\rm i}) + e_{\rm i}(b_{\rm r}d_{\rm r} + b_{\rm i}d_{\rm i}) \mp 2\{e_+(d_{\rm r}e_{\rm i} - d_{\rm i}e_{\rm r}) + e_-(d_{\rm r}e_{\rm r} + d_{\rm i}e_{\rm i})\}] + 4|e|^2|d|^2c_{\rm i} = 0 \tag{41}$$

$$|d|^2\left[e_{\rm r}\left(d_{\rm r}^2 - d_{\rm i}^2\right) + 2e_{\rm i}d_{\rm r}d_{\rm i}\right] \pm 8|e|^2[e_{\rm r}(b_{\rm r}d_{\rm r} + b_{\rm i}d_{\rm i}) + e_{\rm i}(b_{\rm r}d_{\rm i} - b_{\rm i}d_{\rm r}) \pm 2\{e_-(d_{\rm r}e_{\rm i} - d_{\rm i}e_{\rm r}) - e_+(d_{\rm r}e_{\rm r} + d_{\rm i}e_{\rm i})\}] - 4|e|^2|d|^2c_{\rm r} = 0. \tag{42}$$

Using the results of equations (38)–(40*b*), the real and imaginary parts of the eigenvalue can be derived respectively from equations (37*a*) and (37*j*) in the following forms:

$$\begin{aligned} E_{\rm r} = a_{\rm r} \mp \frac{1}{2|e|}[d_{\rm r}e_+ + d_{\rm i}e_-] + \frac{1}{|d|^4}[&e_{\rm r}\{b_{\rm i}(d_{\rm i} + d_{\rm r}) - b_{\rm r}(d_{\rm i} - d_{\rm r})\}\{b_{\rm i}(d_{\rm r} - d_{\rm i}) - b_{\rm r}(d_{\rm r} + d_{\rm i})\} \\ &+ 8\{d_{\rm r}(e_{\rm r} + e_{\rm i}) - d_{\rm i}(e_{\rm r} - e_{\rm i})\}\{d_{\rm r}(e_{\rm i} - e_{\rm r}) - d_{\rm i}(e_{\rm i} + e_{\rm r})\} \\ &\mp 4e_+\{(d_{\rm r}e_{\rm i} - d_{\rm i}e_{\rm r})(b_{\rm i}d_{\rm r} - b_{\rm r}d_{\rm i}) - (d_{\rm r}e_{\rm r} + d_{\rm i}e_{\rm i})(b_{\rm r}d_{\rm r} + b_{\rm i}d_{\rm i})\} \\ &\mp 4e_-\{(d_{\rm r}e_{\rm i} - d_{\rm i}e_{\rm r})(b_{\rm r}d_{\rm r} + b_{\rm i}d_{\rm i}) - (d_{\rm r}e_{\rm r} + d_{\rm i}e_{\rm i})(b_{\rm r}d_{\rm i} - b_{\rm i}d_{\rm r})\} \\ &+ 2e_{\rm i}(b_{\rm i}d_{\rm r} - b_{\rm r}d_{\rm i})(b_{\rm r}d_{\rm r} + b_{\rm i}d_{\rm i})] \end{aligned} \tag{43}$$

$$
\begin{aligned}
E_{\mathrm{i}} = a_{\mathrm{i}} \pm \frac{1}{2|e|}[d_{\mathrm{r}}e_- - d_{\mathrm{i}}e_+] - \frac{1}{|d|^4}\left[e_{\mathrm{i}}\left\{\left(b_{\mathrm{r}}^2 - b_{\mathrm{i}}^2\right)\left(d_{\mathrm{r}}^2 - d_{\mathrm{i}}^2\right) + 4b_{\mathrm{r}}b_{\mathrm{i}}d_{\mathrm{r}}d_{\mathrm{i}}\right\}\right. \\
+ 2e_{\mathrm{r}}(b_{\mathrm{r}}d_{\mathrm{r}} + b_{\mathrm{i}}d_{\mathrm{i}})(b_{\mathrm{i}}d_{\mathrm{r}} - b_{\mathrm{r}}d_{\mathrm{i}}) \mp 4e_+\{(d_{\mathrm{r}}e_{\mathrm{i}} - d_{\mathrm{i}}e_{\mathrm{r}})(b_{\mathrm{r}}d_{\mathrm{r}} + b_{\mathrm{i}}d_{\mathrm{i}}) \\
+ (d_{\mathrm{r}}e_{\mathrm{r}} + d_{\mathrm{i}}e_{\mathrm{i}})(b_{\mathrm{i}}d_{\mathrm{r}} - b_{\mathrm{r}}d_{\mathrm{i}})\} \mp 4e_-\{(d_{\mathrm{r}}e_{\mathrm{i}} - d_{\mathrm{i}}e_{\mathrm{r}})(b_{\mathrm{r}}d_{\mathrm{i}} - b_{\mathrm{i}}d_{\mathrm{r}}) \\
\left. + (d_{\mathrm{r}}e_{\mathrm{r}} + d_{\mathrm{i}}e_{\mathrm{i}})(b_{\mathrm{r}}d_{\mathrm{r}} + b_{\mathrm{i}}d_{\mathrm{i}})\} + 16(d_{\mathrm{r}}e_{\mathrm{r}} + d_{\mathrm{i}}e_{\mathrm{i}})(d_{\mathrm{r}}e_{\mathrm{i}} - d_{\mathrm{i}}e_{\mathrm{r}})\right]. \qquad (44)
\end{aligned}
$$

Next we compute the eigenfunction using (36a) and (36b) and the values of α_{ij} and β_{ij} given in equations (38)–(40). Thus, equation (13) yields the ground-state eigenfunction as

$$
\begin{aligned}
\psi(x_1, p_2) = \exp\left[\frac{|e|}{2|d|^2}\{\{(b_{\mathrm{r}}d_{\mathrm{r}} + b_{\mathrm{i}}d_{\mathrm{i}})e_+ + (b_{\mathrm{r}}d_{\mathrm{i}} - b_{\mathrm{i}}d_{\mathrm{r}})e_- \mp 4(d_{\mathrm{r}}e_{\mathrm{r}} + d_{\mathrm{i}}e_{\mathrm{i}})\} + \mathrm{i}\{(b_{\mathrm{i}}d_{\mathrm{r}} - b_{\mathrm{r}}d_{\mathrm{i}})e_+\right. \\
+ (b_{\mathrm{r}}d_{\mathrm{r}} + b_{\mathrm{i}}d_{\mathrm{i}})e_- \mp 4(d_{\mathrm{r}}e_{\mathrm{i}} - d_{\mathrm{i}}e_{\mathrm{r}})\}\}x \pm \frac{1}{8|e|}\{(d_{\mathrm{r}}e_+ + d_{\mathrm{i}}e_-) \\
\left. - \mathrm{i}(d_{\mathrm{r}}e_- - d_{\mathrm{i}}e_+)\}x^2 \pm \frac{1}{6}(e_+ + \mathrm{i}e_-)x^3\right]. \qquad (45)
\end{aligned}
$$

In what follows we present some special cases of the complex quartic potential (34).

Case 1. First we consider the case of a $\mathcal{PT}$-symmetric potential studied recently by several authors [14, 24]. If we set $a_{\mathrm{i}} = b_{\mathrm{r}} = c_{\mathrm{i}} = d_{\mathrm{r}} = e_{\mathrm{i}} = 0$ in equation (34), the resultant form

$$V(x) = a_{\mathrm{r}} + \mathrm{i}b_{\mathrm{i}}x + c_{\mathrm{r}}x^2 + \mathrm{i}d_{\mathrm{i}}x^3 + e_{\mathrm{i}}x^4 \qquad (46)$$

is analogous to the one studied by Cannata *et al* [24], for $a_{\mathrm{r}} = 0$. In this case, equations (37b)–(37i) reduce to somewhat simpler forms as follows in the same ordering:

$$\alpha_{12}\alpha_{21} = 0 \qquad (47b)$$

$$\alpha_{21}\beta_{11} + \alpha_{11}\alpha_{12} = -\tfrac{1}{4}\, d_{\mathrm{i}} \qquad (47c)$$

$$\alpha_{21}\alpha_{10} - \alpha_{12}\alpha_{01} + \beta_{11}\alpha_{11} = 0 \qquad (47d)$$

$$-\alpha_{12} + \alpha_{11}\alpha_{10} - \beta_{11}\alpha_{01} = \tfrac{1}{4}\, b_{\mathrm{i}} \qquad (47e)$$

$$\alpha_{12}\beta_{11} - \alpha_{21}\alpha_{11} = 0 \qquad (47f)$$

$$\alpha_{21} + \beta_{11}\alpha_{10} + \alpha_{11}\alpha_{01} = 0 \qquad (47g)$$

$$\alpha_{11}^2 - \beta_{11}^2 + 2(\alpha_{01}\alpha_{21} + \alpha_{10}\alpha_{12}) = \tfrac{1}{2}\, c_{\mathrm{r}} \qquad (47h)$$

$$\alpha_{21}^2 - \alpha_{12}^2 = \tfrac{1}{2}\, e_{\mathrm{r}}. \qquad (47i)$$

With regard to the solution of these equations for the ansatz parameters $\alpha_{12}, \alpha_{21}, \alpha_{11}, \beta_{11}, \alpha_{01}, \alpha_{10}$, note from equation (47$b$) that either $\alpha_{12} = 0$ or $\alpha_{21} = 0$ or both are zero. For the cases when either $\alpha_{12} = 0$ or both α_{12} and α_{21} are zero, it can be shown that equations (47b)–(47i) yield $\alpha_{12} = \alpha_{21} = \beta_{11} = \alpha_{11} = \alpha_{10} = \alpha_{01} = 0$. On the other hand, if $\alpha_{21} = 0$ and $\alpha_{12} \neq 0$, equations (47b)–(47i) (except for equation (47e)) can be solved for some negative value of e_{r} (say $e_{\mathrm{r}} = -\bar{e}_{\mathrm{r}}$) in the potential (46). The results obtained are

$$\alpha_{21} = \beta_{11} = \alpha_{01} = 0 \qquad \alpha_{12} = \pm\sqrt{\frac{\bar{e}_{\mathrm{r}}}{2}} \qquad \alpha_{11} = \mp\frac{d_{\mathrm{i}}}{2\sqrt{2\bar{e}_{\mathrm{r}}}} \qquad \alpha_{10} = \pm\frac{\left(4c_{\mathrm{r}}\bar{e}_{\mathrm{r}} - d_{\mathrm{i}}^2\right)}{8\bar{e}_{\mathrm{r}}\sqrt{2\bar{e}_{\mathrm{r}}}} \qquad (48)$$

where $\bar{e}_{\mathrm{r}}$ is real positive. Equation (47e) yields a constraining relation on the potential parameters, namely,

$$8(\bar{e}_{\mathrm{r}})^2 b_{\mathrm{i}} = \mp 16(\bar{e}_{\mathrm{r}})^2\sqrt{2\bar{e}_{\mathrm{r}}} - 4\bar{e}_{\mathrm{r}}\, c_{\mathrm{r}}d_{\mathrm{i}} + d_{\mathrm{i}}^3. \qquad (49)$$

Note in this case that while the imaginary part $E_{\rm i}$ from equation (37*j*) turns out to be zero, the real part of the energy eigenvalue from (37*a*) is given by

$$E_{\rm r} = a_{\rm r} \pm \frac{d_{\rm i}}{\sqrt{2\bar{e}_{\rm r}}} + \frac{1}{64(\bar{e}_{\rm r})^3}\left(4\bar{e}_{\rm r}c_{\rm r} - d_{\rm i}^2\right). \tag{50}$$

Further, $g_{\rm r}$ and $g_{\rm i}$ from equations (36*a*) and (36*b*) also reduce to somewhat simpler forms and finally these results lead the eigenfunction (13) to the form

$$\psi(x_1, p_2) = \exp\left[\tfrac{1}{2}\alpha_{11}x^2 + {\rm i}\left(\alpha_{10}x - \tfrac{1}{3}\alpha_{12}x^3\right)\right] \tag{51}$$

where $\alpha_{11}, \alpha_{12}, \alpha_{10}$ are given in (48) and x in equation (1). Thus, for the potential (46) with the constraining relation (49) and $e_{\rm r}$ negative, the ASE (6) admits the solution (51) with the real eigenvalue given by (50).

Recall that the prescription followed here for the $\mathcal{PT}$-symmetry is of a more general nature than the conventional [6–8, 10–15] one (cf section 1). In our case, this, while affecting the kinetic term in the Hamiltonian, also demands the analyticity of the eigenfunctions $\psi(x)$ through the Cauchy–Riemann conditions (11). In these circumstances, any comparison of the present results with those obtained using the conventional $\mathcal{PT}$-symmetry does not make sense, yet a linkage between the two approaches can be sought, particularly for the real x and p, i.e. by setting $p_2 = x_2 = 0$ in (1) and by obviating the concept of analyticity of $\psi(x)$.

Within the framework of conventional $\mathcal{PT}$-symmetry Cannata *et al* [24] for the potential (46) (with x real and $a_{\rm r} = 0$) arrive only at a three-term constraining relation on the potential parameters whereas in our case it turns out to be a four-term relation (cf equation (49)). Although in our approach the potential (46) appears as a special case of the more general structure (34), the solution corresponding to a real (cf equation (50)) spectrum is obtained only for negative real values of $e_{\rm r}$ in (46), unlike the one discussed by Cannata *et al* for $e_{\rm r} = \pm 1$. Our conclusions, as enumerated above, however, agree with those arrived at by other authors [25, 26] for the conventional $\mathcal{PT}$-symmetry case.

Case 2. Next we analyse the special case of an even power quartic potential, namely,

$$V(x) = a + cx^2 + ex^4 \qquad (a, c, e \text{ complex}) \tag{52}$$

where $b = d = 0$ is set in (34). In this case, the non-repeating equations similar to (37) now turn out to be

$$\left.\begin{array}{lll} E_{\rm r} = a_{\rm r} + 2\left(\alpha_{10}^2 - \alpha_{01}^2\right) & E_{\rm i} = a_{\rm i} - 4\alpha_{10}\alpha_{01} & -4\alpha_{21}\alpha_{12} = e_{\rm i} \\ \alpha_{21}\alpha_{10} - \alpha_{12}\alpha_{01} = \tfrac{1}{4}c_{\rm i} & \alpha_{21}\alpha_{01} + \alpha_{12}\alpha_{10} = \tfrac{1}{4}c_{\rm r} & \alpha_{21}^2 - \alpha_{12}^2 = \tfrac{1}{2}e_{\rm r} \end{array}\right\} \tag{53}$$

and from these equations the ansatz parameters $\alpha_{21}, \alpha_{12}, \alpha_{10}, \alpha_{01}$ can be obtained as before to give

$$\left.\begin{array}{ll} \alpha_{21} = \pm\tfrac{1}{2}e_+ & \alpha_{12} = \mp\tfrac{1}{2}e_- \\ \alpha_{10} = \pm\dfrac{1}{2|e|}(c_{\rm i}e_+ - c_{\rm r}e_-) & \alpha_{01} = \pm\dfrac{1}{2|e|}(c_{\rm r}e_+ + c_{\rm i}e_-) \end{array}\right\} \tag{54}$$

and finally, the energy eigenvalues and the eigenfunction are given by

$$E_{\rm r} = a_{\rm r} + \frac{1}{|e|^2}\left[|e|\left(c_{\rm i}^2 + c_{\rm r}^2\right) - 2e_{\rm i}c_{\rm i}c_{\rm r}\right] \tag{55a}$$

$$E_{\rm i} = a_{\rm i} - \frac{1}{|e|^2}\left[e_{\rm i}\left(c_{\rm i}^2 - c_{\rm r}^2\right) + 2e_{\rm r}c_{\rm i}c_{\rm r}\right] \tag{55b}$$

$$\psi(x) = \exp\left[\pm\frac{1}{6}(e_+ + {\rm i}e_-)x^3 \pm \frac{c}{2|e|}(e_+ - {\rm i}e_-)x\right]. \tag{56}$$

For the case when $a_{\mathrm{i}} = c_{\mathrm{i}} = e_{\mathrm{i}} = 0$ ($\mathcal{PJ}$-symmetric version) in (52), namely,

$$V(x) = a_{\mathrm{r}} + c_{\mathrm{r}} x^2 + e_{\mathrm{r}} x^4 \tag{57}$$

the eigenvalues (55) and the eigenfunction (56) reduce to simple forms as

$$E_{\mathrm{r}} = a_{\mathrm{r}} - \frac{c_{\mathrm{r}}^2}{4e_{\mathrm{r}}} \qquad E_{\mathrm{i}} = 0 \tag{58}$$

$$\psi(x) = \exp\left[\pm\frac{1}{3}\left(\sqrt{\frac{e_{\mathrm{r}}}{2}}x^3 + \frac{3}{2}\,\frac{c_{\mathrm{r}}}{\sqrt{2e_{\mathrm{r}}}}\,x\right)\right]. \tag{59}$$

3.2. Singular potentials

3.2.1. Complex inverse harmonic potential. Consider the case of a simple singular potential,

$$V(x) = \frac{a}{x^2} \qquad (a\ \text{complex}). \tag{60}$$

Note that the potential (60) (of course with real a) is basically a rational solution [27] of the KdV equation (2) in the limit when $U, U', U'' \to 0$ as $|x| \to \infty$ and the same is used here as a potential function in the spirit of equation (3). An ansatz for g_{r} and g_{i} which conform to conditions (15) is chosen as

$$g_{\mathrm{r}}(x_1, p_2) = \beta_{10}\, x_1 - \alpha_{10} p_2 + \beta_1\, \tan^{-1}(x_1/p_2) - \tfrac{1}{2}\,\alpha_1 \ln\left(x_1^2 + p_2^2\right) \tag{61a}$$

$$g_{\mathrm{i}}(x_1, p_2) = \alpha_{10} x_1 + \beta_{10}\, p_2 + \alpha_1\, \tan^{-1}(x_1/p_2) + \tfrac{1}{2}\,\beta_1 \ln\left(x_1^2 + p_2^2\right). \tag{61b}$$

The use of these results in equations (17a) and (17b) and the rationalization of the resultant expressions yields the following set of non-repeating equations:

$$\beta_{10}\,\alpha_{10} = 0 \tag{62a}$$

$$\alpha_{10}\alpha_1 - \beta_{10}\,\beta_1 = 0 \tag{62b}$$

$$a_{\mathrm{i}} + 4\beta_1\,\alpha_1 + 2\beta_1 = 0 \tag{62c}$$

$$\beta_{10}\,\alpha_1 + \alpha_{10}\beta_1 = 0 \tag{62d}$$

$$-2a_{\mathrm{r}} + 4\alpha_1 - 4\beta_1^2 + 4\alpha_1^2 = 0 \tag{62e}$$

with $E_{\mathrm{i}} = E_{\mathrm{r}} = 0$. Clearly, for the unique solution of these equations for the ansatz parameters $\beta_{10}, \alpha_{10}, \alpha_1$ and β_1 one should consider the possibilities (i) $\beta_{10} = 0, \alpha_{10} \neq 0$, (ii) $\alpha_{10} = 0, \beta_{10} \neq 0$ and (iii) $\beta_{10} = \alpha_{10} = 0$. It can be seen that in the first two cases a consistent solution does not exist for the desired a_{r} and a_{i}. However, for case (iii) equations (62a), (62b) and (62d) are trivially satisfied, while equations (62c) and (62e) can be solved in principle for β_1 and α_1. But the value of β_1 from (62c) as $\beta_1 = -a_{\mathrm{i}}/(4\alpha_1 + 2)$ when used in (62e) yields a complicated quartic equation in α_1, namely,

$$4\alpha_1^2(2\alpha_1 + 1)^2 + 4\alpha_1(2\alpha_1 + 1)^2 - 2a_{\mathrm{r}}(2\alpha_1 + 1)^2 - a_{\mathrm{i}}^2 = 0 \tag{63}$$

of which the solution is not very simple. Therefore, for simplicity we set $\alpha_1 = \beta_1$ in equations (62c) and (62e) leading to $\beta_1 = a_{\mathrm{r}}/2$ and a restriction on the potential parameter a_{i} as $a_{\mathrm{i}} < \frac{1}{4}$. For this case the zero energy solution of ASE (6) for the complex singular potential (60) can be expressed as

$$\psi(x_1, p_2) = \left(x_1^2 + p_2^2\right)^{(\mathrm{i}-1)a_{\mathrm{r}}/4} \exp\left[\tfrac{1}{2}\,(1 + \mathrm{i})a_{\mathrm{r}}\, \tan^{-1}(x_1/p_2)\right]. \tag{64}$$

3.2.2. Complex harmonic plus inverse harmonic potential. Here, we consider the potential

$$V(x) = ax^2 + \frac{b}{x^2} \qquad (a, b\,\text{complex}). \tag{65}$$

In this case, the ansatz for g_r and g_i which conform to conditions (15) turns out to be

$$g_\mathrm{r}(x_1, p_2) = \tfrac{1}{2}\alpha_{11}\left(x_1^2 - p_2^2\right) + \beta_{11}x_1p_2 + \beta_1\tan^{-1}(x_1/p_2) - \tfrac{1}{2}\alpha_1\ln\left(x_1^2 + p_2^2\right) \tag{66a}$$

$$g_\mathrm{i}(x_1, p_2) = -\tfrac{1}{2}\beta_{11}\left(x_1^2 - p_2^2\right) + \alpha_{11}x_1p_2 + \alpha_1\tan^{-1}(x_1/p_2) + \tfrac{1}{2}\beta_1\ln\left(x_1^2 + p_2^2\right). \tag{66b}$$

After using the derivatives of these forms of g_r and g_i in equations (17*a*) and (17*b*) and rationalizing the resultant expressions, one obtains the following set of non-repeating equations as before, namely,

$$E_\mathrm{r} = -2\alpha_{11} - 4(\beta_{11}\beta_1 - \alpha_{11}\alpha_1) \tag{67a}$$

$$2\beta_1 + 4\alpha_1\beta_1 = -b_\mathrm{i} \tag{67b}$$

$$2\alpha_1 - 2\beta_1^2 + 2\alpha_1^2 = b_\mathrm{r} \tag{67c}$$

$$2\left(\alpha_{11}^2 - \beta_{11}^2\right) = a_\mathrm{r} \tag{67d}$$

$$4\alpha_{11}\beta_{11} = -a_\mathrm{i} \tag{67e}$$

$$E_\mathrm{i} = 2\beta_{11} - 4(\beta_{11}\alpha_1 + \alpha_{11}\beta_1). \tag{67f}$$

Equations (67*d*) and (67*e*) can be solved in the same way as equations (37*b*) and (37*i*) leading to

$$\alpha_{11} = \pm\tfrac{1}{2}a_+ \qquad \beta_{11} = \mp\tfrac{1}{2}a_-$$

where $a_+ = (|a| + a_\mathrm{r})^{1/2}$, $a_- = (|a| - a_\mathrm{r})^{1/2}$. With regard to the solution of equations (67*b*) and (67*c*), we use the prescription followed earlier for the solutions of equations (62*c*) and (62*e*), namely, the solution of these equations as such leads to a quartic equation of the type (63) in α_1, the solution of which is again a difficult task. Therefore, we assume $\alpha_1 = \beta_1$, as before, in equation (66). For this case, while equation (67*c*) gives $\alpha_1 = \frac{1}{2}b_\mathrm{r}$, the quadratic equation (67*b*) leads to $\alpha_1 = \frac{1}{4}\left[-1 \pm \sqrt{1 - 4b_\mathrm{i}}\right]$. Further, it is noted that the second value of α_1 makes sense only for $b_\mathrm{i} \leqslant \frac{1}{4}$ and the two values of α_1 combined together lead to the constraint on the real and imaginary parts of the parameter b, namely,

$$b_\mathrm{r}^2 + b_\mathrm{r} + b_\mathrm{i} = 0 \tag{68}$$

for the existence of the solution of ASE (6) for the potential (65). For this case the energy eigenvalues from (67*a*) and (67*f*) and the eigenfunction from (13) are given by

$$E_\mathrm{r} = \mp a_+ \pm b_\mathrm{r}(a_+ + a_-) \tag{69a}$$

$$E_\mathrm{i} = \mp a_- \pm b_\mathrm{r}(a_+ - a_-) \tag{69b}$$

$$\psi(x_1, p_2) = \left(x_1^2 + p_2^2\right)^{(\mathrm{i}-1)b_\mathrm{r}/4}\exp\left[\pm\tfrac{1}{4}(a_+ - \mathrm{i}a_-)x^2 + \tfrac{1}{2}b_\mathrm{r}(1+\mathrm{i})\tan^{-1}(x_1/p_2)\right]. \tag{70}$$

It is worth comparing these results with those in equations (28), (29) and (64). Note that while the potential (60) admits the zero energy solutions, nonzero energy solutions are obtained for the potential (65), of course with the constraint (68) on the potential parameters. Further, from (69*b*) it can be noted that $E_\mathrm{i} = 0$ for $b_\mathrm{r} = \pm[(|a| - a_\mathrm{r})/2(|a| - a_\mathrm{i})]^{1/2}$.

Note that in the study of the quantum mechanics of the real version of the potential (65) in one dimension the parameter b is found [28] to take only some discrete values, namely, $b = \frac{1}{2}m(m-1)$, where m is a positive integer, for the existence of a normalizable solution. Here, however, the normalization of $\psi(x_1, p_2)$ involves the integration over the complex x plane and hence will make the situation different. We shall return to some of these details later.

3.3. Exponential potentials

In this category we consider the solution of ASE (6) for the complex Morse potential

$$V(x) = V_0[\mathrm{e}^{-2ax} - 2\mathrm{e}^{-ax}] \qquad (V_0, a \text{ complex}) \tag{71}$$

or, equivalently,

$$\begin{aligned} V_{\mathrm{r}}(x_1, p_2) &= V_{0\mathrm{r}}[\mathrm{e}^{-2X}\cos 2Y - 2\mathrm{e}^{-X}\cos Y] + V_{0\mathrm{i}}[\mathrm{e}^{-2X}\sin 2Y - 2\mathrm{e}^{-X}\sin Y] \\ V_{\mathrm{i}}(x_1, p_2) &= V_{0\mathrm{i}}[\mathrm{e}^{-2X}\cos 2Y - 2\mathrm{e}^{-X}\cos Y] - V_{0\mathrm{r}}[\mathrm{e}^{-2X}\sin 2Y - 2\mathrm{e}^{-X}\sin Y] \end{aligned} \tag{71$'$}$$

where $X = a_{\mathrm{r}}x_1 - a_{\mathrm{i}}p_2$; $Y = a_{\mathrm{i}}x_1 + a_{\mathrm{r}}p_2$; $V_0 = V_{0\mathrm{r}} + \mathrm{i}V_{0\mathrm{i}}$ and $a = a_{\mathrm{r}} + \mathrm{i}a_{\mathrm{i}}$ are used. For the ansatz of the eigenfunction, we take

$$\left.\begin{aligned} g_{\mathrm{r}}(x_1, p_2) &= \beta_1 x_1 - \alpha_1 p_2 + \beta_3\,\mathrm{e}^{-X}\cos Y \\ g_{\mathrm{i}}(x_1, p_2) &= \alpha_1 x_1 + \beta_1 p_2 - \beta_3\,\mathrm{e}^{-X}\sin Y \end{aligned}\right\} \tag{72}$$

which again conform to conditions (15). Using these forms of $V_{\mathrm{r}}, V_{\mathrm{i}}, g_{\mathrm{r}}$ and g_{i} in equations (17*a*) and (17*b*), we rationalize the resultant expressions and obtain the following set of non-repeating equations as before:

$$E_{\mathrm{r}} = 2\left(\alpha_1^2 - \beta_1^2\right) \tag{73a}$$

$$-2V_{0\mathrm{i}} - 4\beta_3 a_{\mathrm{r}} a_{\mathrm{i}} + 4\beta_3(a_{\mathrm{i}}\beta_1 + a_{\mathrm{r}}\alpha_1) = 0 \tag{73b}$$

$$2V_{0\mathrm{r}} - 2\beta_3\left(a_{\mathrm{i}}^2 - a_{\mathrm{r}}^2\right) - 4\beta_3(a_{\mathrm{r}}\beta_1 - a_{\mathrm{i}}\alpha_1) = 0 \tag{73c}$$

$$V_{0\mathrm{i}} - 4a_{\mathrm{i}}a_{\mathrm{r}}\beta_3^2 = 0 \tag{73d}$$

$$-V_{0\mathrm{r}} - 2\beta_3^2\left(a_{\mathrm{i}}^2 - a_{\mathrm{r}}^2\right) = 0 \tag{73e}$$

$$E_{\mathrm{i}} = -4\beta_1\alpha_1. \tag{73f}$$

While equation (73*d*) provides $\beta_3 = \pm(V_{0\mathrm{i}}/4a_{\mathrm{i}}a_{\mathrm{r}})^{1/2}$, equation (73*e*) reduces to a constraining relation among the potential parameters, namely,

$$V_{0\mathrm{i}}\left(a_{\mathrm{i}}^2 - a_{\mathrm{r}}^2\right) + 2V_{0\mathrm{r}}a_{\mathrm{i}}a_{\mathrm{r}} = 0. \tag{74}$$

Alternatively, one can also use (73*e*) to determine β_3 as $\beta_3 = \pm\left[V_{0\mathrm{r}}/2\left(a_{\mathrm{r}}^2 - a_{\mathrm{i}}^2\right)\right]^{1/2}$ and (73*d*) to give the same constraining relation as (74). Further, equations (73*b*) and (73*c*) can be solved for β_{i} and α_{i} to give

$$\left.\begin{aligned} \beta_1 &= \frac{1}{2}a_{\mathrm{r}} + \frac{1}{2\beta_3\,|a|^2}(V_{0\mathrm{i}}a_{\mathrm{i}} + V_{0\mathrm{r}}a_{\mathrm{r}}) \\ \alpha_1 &= \frac{1}{2}a_{\mathrm{i}} + \frac{1}{2\beta_3\,|a|^2}(V_{0\mathrm{i}}a_{\mathrm{r}} - V_{0\mathrm{r}}a_{\mathrm{i}}) \end{aligned}\right\}. \tag{75}$$

Using these results for β_1 and α_1 in (73*a*), (73*f*) and (72), one obtains the expressions for the energy eigenvalues as

$$E_{\mathrm{r}} = -\frac{1}{2}\left(a_{\mathrm{r}}^2 - a_{\mathrm{i}}^2\right) - \frac{V_{0\mathrm{r}}}{\beta_3} - \frac{1}{2\beta_3^2\,|a|^4}\left\{\left(V_{0\mathrm{r}}^2 - V_{0\mathrm{i}}\right)\left(a_{\mathrm{r}}^2 - a_{\mathrm{i}}^2\right) + 4V_{0\mathrm{i}}V_{0\mathrm{r}}a_{\mathrm{i}}a_{\mathrm{r}}\right\} \tag{76a}$$

$$E_\mathrm{i} = -a_\mathrm{i}a_\mathrm{r} - \frac{2V_{0\mathrm{i}}}{\beta_3} + \frac{1}{\beta_3^2\,|a|^4}\left\{V_{0\mathrm{i}}V_{0\mathrm{r}}\left(a_\mathrm{i}^2 - a_\mathrm{r}^2\right) + \left(V_{0\mathrm{r}}^2 - V_{0\mathrm{i}}^2\right)a_\mathrm{i}a_\mathrm{r}\right\} \tag{76b}$$

and for the eigenfunction as

$$\psi(x) = \exp\left[\frac{1}{2}\left(a + \frac{V_0}{\beta_3\,a}\right)x + \beta_3\exp(-ax)\right] \tag{77}$$

where $\beta_3 = \pm\left[V_{0\mathrm{r}}/2\left(a_\mathrm{r}^2 - a_\mathrm{i}^2\right)\right]^{1/2}$. Note that for this choice of β_3 and for $a_\mathrm{i} = V_{0\mathrm{i}} = 0$, expressions (76) and (77) reduce to those obtained for the real V_0, a case in (71) (cf [22]). Further, from (76*b*) a condition among the potential parameters can be derived in this case for the vanishing of E_i.

It may be mentioned that another class of exponential potentials manifesting through the hyperbolic functions has been studied recently by several authors [12, 29]. These potentials are the $\mathcal{PT}$-symmetric ones by construction. While, in general, they admit complex eigenvalues (cf section 1), they are shown to admit the real ones for suitable parametric domains of the potential under study. In this connection the forms of $V(x)$ studied are

$$V(x) = -(\xi\cosh 2x - \mathrm{i}M)^2 \tag{78}$$

by Khare and Mandal [12] and

$$V(x) = -(V_1\,\mathrm{sech}\,x + \mathrm{i}V_2\tanh x)\,\mathrm{sech}\,x \qquad V_1 > 0 \tag{79}$$

by Ahmed [29]. For example, for the potential (79) the discrete eigenvalues are found to be complex-conjugate pairs when $|V_2| > V_1 + \frac{1}{4}$, and real otherwise. Similarly, for the potential (78) the eigenvalues are real for the odd values of the integer M ($M = 1, 3$) and they are complex-conjugate pairs for the even M ($M = 0, 2$). We restrict ourselves from going into further details here.

4. Excited states and orthonormality of eigenfunctions

In this section before discussing the problem of orthonormality of the eigenfunctions corresponding to non-Hermitian Hamiltonians, we demonstrate the viability of the method to study the excited states. In this connection, while we postpone details for a future work [30], the viability of the general prescription of section 2 for the case of excited states is however demonstrated here by way of modifying ansatz (13) to the form

$$\psi(x) = f(x)\exp(g(x)) \tag{80}$$

where $f(x)$ and $g(x)$ are polynomial functions of a complex variable with $f(x) = f_\mathrm{r}(x_1, p_2) + \mathrm{i}f_\mathrm{i}(x_1, p_2)$ and $g(x) = g_\mathrm{r}(x_1, p_2) + \mathrm{i}g_\mathrm{i}(x_1, p_2)$. This form of $\psi(x)$ will replace equations (14) by

$$\left.\begin{aligned}\psi_\mathrm{r}(x_1, p_2) &= \mathrm{e}^{g_\mathrm{r}}(f_\mathrm{r}\cos g_\mathrm{i} - f_\mathrm{i}\sin g_\mathrm{i})\\ \psi_\mathrm{i}(x_1, p_2) &= \mathrm{e}^{g_\mathrm{r}}(f_\mathrm{i}\cos g_\mathrm{i} + f_\mathrm{r}\sin g_\mathrm{i})\end{aligned}\right\}. \tag{81}$$

Now, from equations (81) one immediately obtains the second derivatives as

$$\psi_{\mathrm{r},x_1x_1} = \mathrm{e}^{g_\mathrm{r}}(B\cos g_\mathrm{i} - A\sin g_\mathrm{i}) \qquad \psi_{\mathrm{i},x_1x_1} = \mathrm{e}^{g_\mathrm{r}}(A\cos g_\mathrm{i} + B\sin g_\mathrm{i}) \tag{82}$$

where

$$\begin{aligned}A = {} & f_{\mathrm{i},x_1x_1} - f_\mathrm{i}\left(g_{\mathrm{i},x_1}\right)^2 + f_\mathrm{i}\left(g_{\mathrm{r},x_1}\right)^2 + 2f_{\mathrm{r},x_1}g_{\mathrm{i},x_1} + 2f_{\mathrm{i},x_1}g_{\mathrm{r},x_1}\\ & + 2f_\mathrm{r}g_{\mathrm{r},x_1}g_{\mathrm{i},x_1} + f_\mathrm{r}g_{\mathrm{i},x_1x_1} + f_\mathrm{i}g_{\mathrm{r},x_1x_1}\\ B = {} & f_{\mathrm{r},x_1x_1} - f_\mathrm{r}\left(g_{\mathrm{i},x_1}\right)^2 + f_\mathrm{r}\left(g_{\mathrm{r},x_1}\right)^2 - 2f_{\mathrm{i},x_1}g_{\mathrm{i},x_1} + 2f_{\mathrm{r},x_1}g_{\mathrm{r},x_1}\\ & - 2f_\mathrm{i}g_{\mathrm{r},x_1}g_{\mathrm{i},x_1} - f_\mathrm{i}g_{\mathrm{i},x_1x_1} + f_\mathrm{r}g_{\mathrm{r},x_1x_1}.\end{aligned}$$

The use of these results in ($12a'$) and ($12b'$) yields a pair of coupled PDEs, namely,

$$E_{\mathrm{r}} = V_{\mathrm{r}} - \frac{2}{f_{\mathrm{r}}^2 + f_{\mathrm{i}}^2}(f_{\mathrm{r}}B + f_{\mathrm{i}}A) \qquad E_{\mathrm{i}} = V_{\mathrm{i}} - \frac{2}{f_{\mathrm{r}}^2 + f_{\mathrm{i}}^2}(f_{\mathrm{r}}A - f_{\mathrm{i}}B) \tag{83}$$

which in turn, after substituting the expressions for A and B, give rise to

$$\begin{aligned} E_{\mathrm{r}} = V_{\mathrm{r}} - 2\left[g_{\mathrm{r},x_1x_1} - \left(g_{\mathrm{i},x_1}\right)^2 + \left(g_{\mathrm{r},x_1}\right)^2\right] - \frac{2}{f_{\mathrm{r}}^2 + f_{\mathrm{i}}^2}\big[f_{\mathrm{r}}\left(f_{\mathrm{r},x_1x_1} + 2f_{\mathrm{r},x_1}g_{\mathrm{r},x_1} - 2f_{\mathrm{i},x_1}g_{\mathrm{i},x_1}\right) \\ + f_{\mathrm{i}}\left(f_{\mathrm{i},x_1x_1} + 2f_{\mathrm{r},x_1}f_{\mathrm{i},x_1} + 2f_{\mathrm{i},x_1}g_{\mathrm{r},x_1}\right)\big] \end{aligned} \tag{84a}$$

$$\begin{aligned} E_{\mathrm{i}} = V_{\mathrm{i}} - 2\left[g_{\mathrm{i},x_1x_1} + 2g_{\mathrm{r},x_1}g_{\mathrm{i},x_1}\right] - \frac{2}{f_{\mathrm{r}}^2 + f_{\mathrm{i}}^2}\big[f_{\mathrm{r}}\left(f_{\mathrm{i},x_1x_1} + 2f_{\mathrm{r},x_1}g_{\mathrm{i},x_1} + 2f_{\mathrm{i},x_1}g_{\mathrm{r},x_1}\right) \\ + f_{\mathrm{i}}\left(-f_{\mathrm{r},x_1x_1} + 2f_{\mathrm{i},x_1}g_{\mathrm{i},x_1} - 2f_{\mathrm{r},x_1}g_{\mathrm{r},x_1}\right)\big]. \end{aligned} \tag{84b}$$

These are the equations to be rationalized to obtain the excited states for a given potential in the same way as we have done in the previous sections with equations (17*a*) and (17*b*). Note that in equations (84*a*) and (84*b*), while other terms conform to the results for the ground state (cf equations (17*a*) and (17*b*)), the last term in these equations is the contribution of the additional factor $f(x)$ in ansatz (80). It can be immediately seen that this contribution vanishes for the case when $f(x) = $ constant, i.e. for the ground state. Some explicit applications of equations (84*a*) and (84*b*) for a complex sextic potential are demonstrated elsewhere [30].

Next we comment here on the question of normalization of the eigenfunctions for the non-Hermitian Hamiltonian operators. For the (conventional) $\mathcal{PT}$-symmetric potentials, however, the issue of normalization of the eigenstates has been addressed by Bender and Turbiner [6], Bender and Boettcher [7], Bender *et al* [31] and more recently by Ahmed [29] and Bagchi *et al* [32]. In the approach of Bender and his co-workers [6, 7, 25, 31] the eigenstates for the (conventional) $\mathcal{PT}$-symmetric Hamiltonians which are complex, well behaved in $(-\infty, \infty)$ and asymptotically vanishing on the real line, are normalizable. As a matter of fact, in this case the real x is replaced by a contour in the complex plane along which the Schrödinger differential equation holds and subsequently the imposed boundary conditions lead to quantization at the end points of the contour via a WKB-type approach. Further, for the regions in the cut complex x-plane (where $\psi(x)$ vanishes asymptotically as $|x| \longrightarrow \infty$) Bender *et al* have used [33] the concept of wedges bounded by Stokes lines in their example-based discussions.

Now the question arises: what prescription in general one should use for the normalization and orthogonality of the eigenstates corresponding to a non-Hermitian operator? In this connection the prescription of Ahmed [29] appears more appealing. Ahmed has studied the discrete spectrum for the $\mathcal{PT}$-symmetric potential (79) and the orthogonality of states $\psi_1(x)$, $\psi_2(x)$ corresponding to eigenvalues E_1 and E_2 is defined by [29]

$$\int_{-\infty}^{\infty} \psi_1(x)\psi_2(x)\,\mathrm{d}x = 0 \tag{85}$$

for $E_1 \neq E_2$. Note the absence of the complex conjugation in (85). Further, in view of the $\mathcal{PT}$-operations involved in the method a more general condition suggested by Ahmed is

$$\int_{-\infty}^{\infty} {\psi_1}^{\mathcal{PT}}(x)\psi_2(x)\,\mathrm{d}x = 0 \tag{86}$$

for $E_1^* \neq E_2$, i.e. for the case of broken $\mathcal{PT}$-symmetry. Here $\psi^{\mathcal{PT}}(x) = \psi^*(-x)$. For the potential (79) both the above prescriptions are found to hold in general within the framework of the underlying constraining relations among the potential parameters. A basis to these conditions of orthogonality is also sought by Bagchi *et al* [32] in the equation of continuity.

In the present work, although we have restricted ourselves to the ground-state solution of ASE (6), yet their normalization and, subsequently for the case of excited state solutions, the study of orthogonality of the complex solutions is desirable. Note that our computed eigenfunctions for various complex potentials (cf equations (26), (29), (33), (45), (51), (56), (64), (70), (77)) basically are the complex functions of the complex variable x (cf equation (1)). Therefore, the normalization constant N in $\psi(x) = N\exp(g(x))$ (or for the excited states in $\psi(x) = Nf(x)\exp(g(x))$), in general, need not be a real number and the same should be determined from the contour integral

$$\int \psi^2(x)\,\mathrm{d}x = 1 \qquad \text{or} \qquad N = \left(\int \mathrm{e}^{2g(x)}\,\mathrm{d}x\right)^{-1/2}. \tag{87}$$

Unfortunately, the derived eigenfunctions in the present method do not exhibit any pole by construction, except for the case of singular potentials (cf equations (64) and (70) in which for certain values of a_r or b_r, $\psi(x)$ can have poles). Thus, according to the Cauchy residue theorem the integral in (87) vanishes, leaving behind the question of normalization of $\psi(x)$. On the other hand, if we proceed via a two-real-dimension analogue [3, 4] of the one complex dimension in view of definition (1), the integral (87) can be recast in the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi^2(x_1, p_2)\,\mathrm{d}x_1\,\mathrm{d}p_2 = 1$$

or

$$N^{-2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[2\{g_\mathrm{r}(x_1, p_2) + \mathrm{i}g_\mathrm{i}(x_1, p_2)\}]\,\mathrm{d}x_1\,\mathrm{d}p_2 \tag{88}$$

and the complex N can be determined. In the same vein, in the present approach, one can introduce the orthogonality of the eigenfunctions $\psi_1(x)$ and $\psi_2(x)$ corresponding to the complex eigenenergies E_1 and E_2 as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_1(x_1, p_2)\psi_2(x_1, p_2)\,\mathrm{d}x_1\,\mathrm{d}p_2 = 0 \tag{89}$$

for $|E_1| \neq |E_2|$. This is a rather strong condition for the orthogonality of ψ_1 and ψ_2. However, other weak conditions for the validity of (89) could be for (i) $E_1 \neq E_2$, (ii) $E_1^* \neq E_2$ or $E_1 \neq E_2^*$, depending upon the nature of the potential.

In conventional (Hermitian) quantum mechanics (CQM) the use of boundary conditions and the normalization of the eigenfunction have some physical bearing in the sense that these features of the wavefunction are meant to fix the geometry of the quantum system. In particular, the boundary conditions will help in eliminating one of the linearly independent solutions out of the general solution (which is a linear combination of two linearly independent solutions) of the second-order Schrödinger wave equation, the normalization of the eigenfunction, on the other hand, ensures the probability of finding the particle within those boundaries. Further, with regard to the mathematical content of the eigenfunction it is considered as a complex function of the real variables and the complexity of the same arises mainly from the angular part of the total wavefunction. In the present analogous (non-Hermitian) quantum mechanics (AQM) described by ASE (6) the situation is however different. In some sense AQM is equivalent [4] to studying CQM in two real dimensions x_1, p_2. Although the role of these two dimensions manifests clearly in the eigenfunction, however, it reduces to that of one dimension for its derivatives in view of condition (15). That is why equations (17*a*) and (17*b*), after their rationalization, yield identical equations for a given potential $V(x)$.

With regard to the boundary conditions on $\psi(x)$ in the present approach, it is interesting to note that all the computed eigenfunctions for the bound states (i.e. with the negative sign in the exponent), namely, (26), (29), (33), (45), (51), (56) and (70), obey the condition $\lim_{|x|\longrightarrow\infty} \psi(x) = 0$ even for x defined in (1). This is in conformity with what has been emphasized by Bender *et al* [25] for the asymptotic solutions. It can be seen that with some restrictions on the potential parameters the eigenfunction (77) corresponding to potential (71) also fulfils this requirement. However, the pure singular potential (60) exhibits only the scattering state solutions (64).

5. Summary and discussion

With a view to exploring new vistas with regard to the nature of complex spectra and associated eigenfunctions for the non-Hermitian Hamiltonian operators, the quasi-exact solutions of ASE (6) are investigated for a variety of complex potential functions. In particular, the ground-state solutions of equation (6) are obtained for power, singular and exponential potentials. Besides the complexity of the phase space produced by (1), the complexity of the parameters of potential $V(x)$ is also considered. It is this latter consideration which is found to suggest the nonvanishing of the imaginary part of the eigenspectrum in most cases. While several variants of the complex oscillator potential (including the shifted oscillator) and complex quartic potential (including the even power and $\mathcal{PJ}$-symmetric ones) are investigated, a complex 'pure' singular potential (cf equation (60)) is found to admit only zero energy solutions as is the case with real singular potentials. Introduction of a complex harmonic piece (cf equation (65)) in (60), however, leads to nonzero energy solutions. It may be emphasized that the solutions of ASE (6) in some of the above-mentioned cases are obtained only in the presence of certain constraining relation(s) among the potential parameters, namely, equations (41) and (42) for the general quartic case (34), equation (49) for the $\mathcal{PJ}$-symmetric quartic case (46), and equation (68) for the harmonic plus inverse harmonic potential (65). The solution for the complex Morse potential (71) is obtained when the real and imaginary parts of the parameters V_0 and a in (71) satisfy the constraining relation (74).

On the basis of the above studies and those carried out in [22] the following general remarks are in order:

(1) In the present framework of the extended complex phase space produced by (1) the imaginary part of the eigenvalue always vanishes for the solvable cases of ASE (6) as long as all the parameters of the complex potential $V(x)$ are real. In this respect the results obtained in the present approach coincide with those derived by demanding the invariance of the given Hamiltonian under the $\mathcal{PJ}$-operation. The present approach, in this case, provides results without any constraint on the potential parameters, the conventional $\mathcal{PJ}$-symmetry approach (used extensively [6–15] in recent years), however, yields real eigenvalues only in a limited parametric domain.

(2) In spite of the fact that a physical basis for some of the steps in the present approach (such as the orthogonality and the completeness of states) has yet to be explored, it is quite general and viable in the sense that (i) the analyticity property of the eigenfunction greatly simplifies the underlying computation in determining the nature of the spectra, (ii) a simple extension of the parameters from the real to the complex domain immediately yields the complex spectrum, at least for solvable cases, and (iii) for a solvable case, the constraining relations (if they are there at all) immediately help in identifying the usable domain for the parameters in the potential function $V(x)$ which in turn would suggest the desired features in the spectrum.

(3) The $\mathcal{PJ}$-symmetric potentials studied extensively [6–15] deal mainly with the complexity arising from the potential parameters in their restricted domains for a real eigenvalue

spectra. In this respect this approach could also be considered as a special case of the present general method (cf section 3.1.2, case 1). To demonstrate this fact note that for real x, when the solution (51) is substituted in equations (6), the rationalization of the resultant expression immediately yields the same results as obtained by Bender *et al* [25] for $J = 1$ in their $\mathcal{PJ}$-symmetric potential. On the other hand, the real eigenvalue spectra and the constraining relation obtained for the potential (46) in the present approach (cf equations (49) and (50)) have a basis in the analyticity property of $\psi(x)$ and the complex nature of the underlying phase space.

The project initiated in this paper is not yet over. From the point of view of physical applications of the results derived here, several aspects of the present method of handling the non-Hermitian operators in quantum mechanics need to be explored further, particularly in the light of the newly introduced [34] concept of pseudo-Hermiticity for a complex Hamiltonian. Such studies are in progress.

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